

RESTRICTING SCHUBERT CLASSES

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Introduction

The goal of the present note is twofold. Firstly, we correct some points in the paper (quoted in the following as [P]):

P. Pragacz: “A generalization of the Macdonald-You formula”, *Journal of Algebra* **204**, 573–587 (1998).

Secondly, by comparing the formula appearing in the title of [P] with results of Stembridge [St] and some other combinatorial results, we deduce some new identities in Propositions 2–6. They concern: restrictions of Schubert classes to the cohomology of Lagrangian Grassmannians as well as relations between Q -functions, Stembridge’s coefficients, and various “hook numbers”. Also, we provide some examples illustrating [P] and the formulas given in the present note.

In this note, any unexplained notation or quotation stems from [P]. However, in order to make the notation maximally compatible with that used in [St] (which is our principal reference here), we label strict partitions by λ , and μ usually denotes an ordinary partition, contrariwise to [P].

1. Erratum to [P]

Due to some bugs in the computer system SCHUR [Sch], [P, Example 3(b)] was miscalculated: the quadratic expression in Q -functions displayed there, written as a \mathbb{Z} -linear combination of Q -functions, contains *no* negative summands. Consequently the sentence on p.585, lines 6–7 from the bottom, is to be withdrawn from [P]. (These corrections *do not* affect other results of [P], in particular the main formulas.)

2. Nonnegativity of the restriction coefficients

In fact, if

$$(1) \quad i^*(\sigma_\mu) = \sum_{\lambda} c_{\lambda\mu} \sigma'_\lambda,$$

with $c_{\lambda\mu} \in \mathbb{Z}$, then all the coefficients $c_{\lambda\mu}$ are nonnegative. Perhaps the easiest way to see this, is the following. Let for $a \in H^*(G; \mathbb{Z})$, $\int_G a$ stand for the degree of the

top codimensional component of a , and define similarly $\int_{G'} b$ for $b \in H^*(G'; \mathbb{Z})$. Given a strict partition $\lambda \subset (n, n-1, \dots, 1)$, we denote by λ^\vee the strict partition whose parts complement those of λ in $\{1, \dots, n\}$. We record the following property [P2]:

Lemma 1 (Duality). *The basis $\{\sigma'_\lambda\}$ of the group $H^{2p}(G'; \mathbb{Z})$ and the basis $\{\sigma'_{\lambda^\vee}\}$ of the group $H^{n(n+1)-2p}(G'; \mathbb{Z})$ are dual under the pairing $(a, b) \mapsto \int_{G'} a \cdot b$ of Poincaré duality.*

Now, if $i^*(\sigma_\mu) = \sum_\lambda c_{\lambda\mu} \sigma'_\lambda$, with $c_{\lambda\mu} \in \mathbb{Z}$, then it follows from the duality property that

$$(2) \quad c_{\lambda\mu} = \int_{G'} i^*(\sigma_\mu) \cdot \sigma'_{\lambda^\vee}.$$

Using the projection formula for i , this is rewritten as

$$(3) \quad c_{\lambda\mu} = \int_G \sigma_\mu \cdot i_*(\sigma'_{\lambda^\vee}).$$

Regard G as a homogeneous space $GL(V)/P$, where P is a suitable parabolic subgroup of $GL(V)$. Let $\Omega \subset G$ be a Schubert variety representing σ_μ and let $\Omega' \subset G' \subset G$ be a Schubert variety representing σ'_{λ^\vee} . Using e.g. Kleiman's theorem on a general translate [K], we can replace Ω by a translate by an element $g \in GL(V)$ such that $g \cdot \Omega$ and Ω' meet properly, and this intersection is represented as a nonnegative zero-cycle. This shows that $c_{\lambda\mu} \geq 0$.

A similar property holds in the following more general setting. Let now $G \supset P \supset B$ be a semisimple linear algebraic group, a parabolic subgroup, and a Borel subgroup. In a generalized flag variety G/P , one has Schubert varieties BwP/P and their Schubert classes in $H^*(G/P; \mathbb{Z})$ indexed by a corresponding subset of the Weyl group. These Schubert classes enjoy a similar duality property. In an analogous way, using a general translate argument, one shows that the fundamental class of any subscheme of G/P is a \mathbb{Z} -linear combination of the Schubert classes in $H^*(G/P; \mathbb{Z})$ with nonnegative coefficients. Combining this with a well-known fact about pulling back the class of a Cohen-Macaulay subscheme (see, e.g., Lemma on p.108 in [F-P]), we get the following result (also implying the nonnegativity of the above $c_{\lambda\mu}$):

Proposition 1. *Let $f : G/P \rightarrow Y$ be morphism to a nonsingular variety Y . Let Z be a pure-dimensional closed Cohen-Macaulay subscheme of Y . Then $f^*([Z])$ is a \mathbb{Z} -linear combination of the Schubert classes in $H^*(G/P; \mathbb{Z})$ with nonnegative coefficients.*

3. Stembridge's coefficients

We recall (see the discussion after [P, Proposition 8]) that the coefficients appearing in (1) and those appearing in:

$$(4) \quad \eta(s_\mu) = \sum g_{\lambda\mu} Q_\lambda$$

satisfy $c_{\lambda\mu} = g_{\lambda\mu}$. Here, we take sufficiently large Grassmannians $i : G' \hookrightarrow G$. To be more precise, this means that given μ , we take $n \geq |\mu|$ so that any strict partition λ with $|\lambda| = |\mu|$ is contained in $(n, n-1, \dots, 1)$. Consequently, all the coefficients $g_{\lambda\mu}$ are nonnegative. But this result, together with a combinatorial interpretation of the $g_{\lambda\mu}$'s, was already established by Stembridge in [St].¹ Indeed, the last displayed (unnumbered) equality before [St, Theorem 9.3]:

$$(5) \quad "S_\mu = \sum_{\lambda \in DP_n} g_{\lambda\mu} Q_\lambda"$$

is identical with (4) because S_μ in the notation of [St] (and [M]) is equal to $\eta(s_\mu)$ in our notation.² In [St], (5) is a consequence of the equality

$$(6) \quad P_\lambda = \sum_{|\mu|=|\lambda|} g_{\lambda\mu} s_\mu,$$

where $P_\lambda = 2^{-l(\lambda)} Q_\lambda$, and comparison of the canonical scalar products on the ring of all symmetric functions with that on the ring of Q -functions. To the nonnegativity of $g_{\lambda\mu}$ is given, in loc. cit., several interpretations in representation theory, some of which go back to Morris and Stanley.

(Observe that (4) and (6) yield the following expression for $\eta(P_\lambda)$:

$$(7) \quad \eta(P_\lambda) = \sum_{|\mu|=|\lambda|} g_{\lambda\mu} \eta(s_\mu) = \sum_{|\mu|=|\lambda|} \sum_{|\nu|=|\lambda|} g_{\lambda\mu} g_{\nu\mu} Q_\nu.$$

Stembridge [St] also established a combinatorial interpretation of the numbers $f_{\mu\nu}^\lambda$ appearing as coefficients in the expansion:

$$(8) \quad P_\mu P_\nu = \sum_{\lambda} f_{\mu\nu}^\lambda P_\lambda,$$

where μ , ν , and λ denote now strict partitions. It will be convenient to set

$$(9) \quad e_{\mu\nu}^\lambda := 2^{l(\mu)+l(\nu)-l(\lambda)} f_{\mu\nu}^\lambda.$$

There exists a geometric analogue of (8): in the cohomology ring $H^*(G'; \mathbb{Z})$ of a sufficiently large Lagrangian Grassmannian,

$$(10) \quad \sigma'_\mu \cdot \sigma'_\nu = \sum_{\lambda} e_{\mu\nu}^\lambda \sigma'_\lambda.$$

(See [P2, Sect.6].)

Stembridge's combinatorial description of the above $f_{\mu\nu}^\lambda$ and $g_{\lambda\mu}$ can be summarized by the following:

¹The fact that this result was already established by Stembridge, has been learned by the author only in June 1999.

²Notice that this result, established in [St], had been independently established in [M].

Theorem [St]. (i) The coefficient $f_{\mu\nu}^\lambda$ is equal to the number of marked shifted tableaux T of shape λ/μ and weight (or content) ν such that:

- (a) The word $w(T)$ associated with T ([St, Sect.8] and [M, p.258]) has the lattice property in the sense of loc.cit.;
- (b) for each $k \geq 1$, the rightmost occurrence of k' in $w(T)$ precedes the last occurrence of k .

(ii) The coefficient $g_{\lambda\mu}$ is equal to the number of unshifted marked tableaux T of shape μ and weight λ satisfying (a) and (b) above.

For all unexplained here combinatorial notions, we refer the reader to [St, Sect.6 and 8], [P2, Sect.4], and to [M, III.8 pp.255–259]. We make no attempt to make a complete survey here. Some examples of the coefficients $g_{\lambda\mu}$ will be given below.

Summarizing the content of this section, we record:

Proposition 2. We have for a partition $\mu \subset (n^n)$

$$(11) \quad i^*(\sigma_\mu) = \sum_{\lambda} g_{\lambda\mu} \sigma'_\lambda,$$

where λ runs over strict partitions contained in $(n, n-1, \dots, 1)$, and $g_{\lambda\mu}$ is the Stembridge coefficient described in Theorem (ii).

4. Quadratic relations between Q -functions

We pass now to some applications of the generalized Macdonald-You formula ([L-L2], [P, Corollary 2]):

$$(12) \quad 2^n \eta(s_\mu) = \sum Q_{(a_{i_1}, \dots, a_{i_k})} \cdot Q_{A \# B \setminus (a_{i_1}, \dots, a_{i_k})}.$$

Recall that here, for $\mu = (\alpha_1, \dots, \alpha_n | \beta_1, \dots, \beta_n)$ in Frobenius notation,

$$(13) \quad A = (a_1, \dots, a_n) := (\alpha_1 + 1, \dots, \alpha_n + 1), \quad B := (\beta_1, \dots, \beta_n),$$

and the sum is over all sequences $1 \leq i_1 < \dots < i_k \leq n$ and $k = 0, 1, \dots, n$.

Since $\eta(e_i) = \eta(h_i)$, where h_i is the i th complete homogeneous symmetric function, we have for a partition μ

$$(14) \quad \eta(s_{\mu^\sim}) = \eta(s_\mu),$$

where $\mu^\sim = (\beta_1, \dots, \beta_n | \alpha_1, \dots, \alpha_n)$ is the conjugate partition of μ . We set in addition

$$(15) \quad C = (c_1, \dots, c_n) := (\beta_1 + 1, \dots, \beta_n + 1), \quad D := (\alpha_1, \dots, \alpha_n).$$

Then (12) and (14) imply the following

Proposition 3. *We have*

$$(16) \quad \sum Q_{(a_{i_1}, \dots, a_{i_k})} \cdot Q_{A \# B \setminus (a_{i_1}, \dots, a_{i_k})} = \sum Q_{(c_{i_1}, \dots, c_{i_k})} \cdot Q_{C \# D \setminus (c_{i_1}, \dots, c_{i_k})},$$

where the sums are over all sequences $1 \leq i_1 < \dots < i_k \leq n$ and $k = 0, 1, \dots, n$.

The relations (16), regarded from the side of Q -functions, seem to be rather nontrivial. For instance, for $\mu = (5^3 31^3) = (432|621)$, so $A = (5, 4, 3)$, $B = (6, 2, 1)$, $C = (7, 3, 2)$, and $D = (4, 3, 2)$, we get the equation

$$(17) \quad \begin{aligned} & Q_{654321} - Q_5 \cdot Q_{64321} + Q_4 \cdot Q_{65321} - Q_3 \cdot Q_{65421} - Q_{54} \cdot Q_{6321} \\ & + Q_{53} \cdot Q_{6421} - Q_{43} \cdot Q_{6521} + Q_{543} \cdot Q_{621} \\ & = Q_{32} \cdot Q_{7432} + Q_{732} \cdot Q_{432}. \end{aligned}$$

Using a (hopefully) debugged version of SCHUR, (16) is expressed as the following \mathbb{Z} -linear combination of the Q_λ 's :

$$\begin{array}{lllll} 8Q_{11 \ 64} & + 8Q_{11 \ 631} & + 8Q_{11 \ 541} & + 8Q_{11 \ 532} & + 8Q_{10 \ 74} \\ + 8Q_{10 \ 731} & + 8Q_{10 \ 65} & + 24Q_{10 \ 641} & + 24Q_{10 \ 632} & + 24Q_{10 \ 542} \\ + 8Q_{10 \ 5321} & + 8Q_{975} & + 16Q_{9741} & + 16Q_{9732} & + 16Q_{9651} \\ + 48Q_{9642} & + 16Q_{96321} & + 16Q_{9543} & + 16Q_{95421} & + 8Q_{8751} \\ + 24Q_{8742} & + 8Q_{87321} & + 24Q_{8652} & + 24Q_{8643} & + 24Q_{86421} \\ + 8Q_{85431} & + 8Q_{7653} & + 8Q_{76521} & + 8Q_{76431} & \end{array}$$

Consequently, taking sufficiently large Grassmannians $i : G' \hookrightarrow G$, we have

$$\begin{array}{lllll} i^*(\sigma_{5^3 31^3}) = & & & & \\ \sigma'_{11 \ 64} & + \sigma'_{11 \ 631} & + \sigma'_{11 \ 541} & + \sigma'_{11 \ 532} & + \sigma'_{10 \ 74} \\ + \sigma'_{10 \ 731} & + \sigma'_{10 \ 65} & + 3\sigma'_{10 \ 641} & + 3\sigma'_{10 \ 632} & + 3\sigma'_{10 \ 542} \\ + \sigma'_{10 \ 5321} & + \sigma'_{975} & + 2\sigma'_{9741} & + 2\sigma'_{9732} & + 2\sigma'_{9651} \\ + 6\sigma'_{9642} & + 2\sigma'_{96321} & + 2\sigma'_{9543} & + 2\sigma'_{95421} & + \sigma'_{8751} \\ + 3\sigma'_{8742} & + \sigma'_{87321} & + 3\sigma'_{8652} & + 3\sigma'_{8643} & + 3\sigma'_{86421} \\ + \sigma'_{85431} & + \sigma'_{7653} & + \sigma'_{76521} & + \sigma'_{76431} & \end{array}$$

So e.g. we have: $g_{(5^3 31^3) \ (11 \ 64)} = 1$, $g_{(5^3 31^3) \ (10 \ 641)} = 3$, $g_{(5^3 31^3) \ (9741)} = 2$, and $g_{(5^3 31^3) \ (9642)} = 6$.

5. Linear relations between Stembridge's coefficients

Combining (4), (12), and (16), we have in the above notation, associated with a fixed μ

$$(18) \quad \begin{aligned} & \sum Q_{(a_{i_1}, \dots, a_{i_k})} \cdot Q_{A \# B \setminus (a_{i_1}, \dots, a_{i_k})} \\ & = \sum Q_{(c_{i_1}, \dots, c_{i_k})} \cdot Q_{C \# D \setminus (c_{i_1}, \dots, c_{i_k})} = 2^n \sum_{\lambda} g_{\lambda \mu} Q_{\lambda}, \end{aligned}$$

where the first two sums are over all sequences $1 \leq i_1 < \dots < i_k \leq n$ and $k = 0, 1, \dots, n$.

The equalities (18) imply linear relations between the $e_{\mu\nu}^\lambda$'s and $g_{\lambda\mu}$'s. Given a sequence of different positive integers $K = (k_1, \dots, k_l)$, there is a permutation $w = w_K \in S_l$ such that $k_{w(1)} > \dots > k_{w(l)} > 0$. Denote this last-mentioned strict partition by $< K >$. Then given strict partitions μ, λ and a sequence K as above, we set

$$(19) \quad e_{\mu \ K}^\lambda := \text{sgn}(w_K) e_{\mu \ < K >}^\lambda.$$

From (18) and (16) we get the following result:

Proposition 4. *For a fixed partition μ and strict partition λ with $|\mu| = |\lambda|$, we have in the above notation associated with μ*

$$(20) \quad \begin{aligned} 2^n g_{\lambda\mu} &= \sum e_{(a_{i_1}, \dots, a_{i_k}), A \# B \setminus (a_{i_1}, \dots, a_{i_k})}^\lambda \\ &= \sum e_{(c_{i_1}, \dots, c_{i_k}), C \# D \setminus (c_{i_1}, \dots, c_{i_k})}^\lambda \end{aligned}$$

where the sums are over all sequences $1 \leq i_1 < \dots < i_k \leq n$ for which $A \# B \setminus (a_{i_1}, \dots, a_{i_k})$ (resp. $C \# D \setminus (c_{i_1}, \dots, c_{i_k})$) is a sequence of different integers, and $k = 0, 1, \dots, n$.

For instance, for any strict partition λ with $|\lambda| = 21$, and for $\mu = (5^3 31^3) = (432|621)$, we get the equations:

$$\begin{aligned} 2^3 g_{\lambda (5^3 31^3)} &= e_{(654321) (\emptyset)}^\lambda - e_{(5) (64321)}^\lambda + e_{(4) (65321)}^\lambda - e_{(3) (65421)}^\lambda - e_{(54) (6321)}^\lambda \\ &\quad + e_{(53) (6421)}^\lambda - e_{(43) (6521)}^\lambda + e_{(543) (621)}^\lambda \\ &= e_{(32) (7432)}^\lambda + e_{(732) (432)}^\lambda. \end{aligned}$$

6. The class $i_*(\sigma'_\lambda)$ as a \mathbb{Z} -linear combination of the σ_μ 's

By reasoning similarly as in §4, one shows that for any proper morphism $f : X \rightarrow G/P$ from a scheme X to a generalized flag variety G/P , and for any irreducible subscheme $Y \subset X$, $f_*([Y])$ is a \mathbb{Z} -linear combination of Schubert classes in $H^*(G/P; \mathbb{Z})$ with nonnegative coefficients.

The next proposition will give $i_*(\sigma'_\lambda)$ as an explicit \mathbb{Z} -linear combination of the σ_μ 's. Given a partition $\mu \subset (n^n)$, we set $\mu^* := (n - \mu_n, \dots, n - \mu_1)$. The following duality property is a well-known result of Schubert calculus [F]:

Lemma 2. *The basis $\{\sigma_\mu\}$ of the group $H^{2p}(G; \mathbb{Z})$ and the basis $\{\sigma_{\mu^*}\}$ of the group $H^{2(n^2-p)}(G; \mathbb{Z})$ are dual under the pairing $(a, b) \mapsto \int_G a \cdot b$ of Poincaré duality.*

We now state:

Proposition 5. *For a fixed strict partition $\lambda \subset (n, n-1, \dots, 1)$, we have*

$$(21) \quad i_*(\sigma'_\lambda) = \sum_{|\mu|=|\lambda|+n(n-1)/2} g_{\lambda^\vee, \mu^*} \sigma_\mu,$$

where μ runs over partitions contained in (n^n) and g_{λ^\vee, μ^*} is the Stembridge coefficient described in Theorem (ii).

Indeed, if $i_*(\sigma'_\lambda) = \sum_\mu m_{\lambda\mu} \sigma_\mu$, with $m_{\lambda\mu} \in \mathbb{Z}$ (so that $|\mu| = |\lambda| + n(n-1)/2$), then it follows from Lemma 2 that

$$(22) \quad m_{\lambda\mu} = \int_G (i_* \sigma'_\lambda) \cdot \sigma_{\mu^*}.$$

Using the projection formula for i , this is rewritten as

$$(23) \quad m_{\lambda\mu} = \int \sigma'_\lambda \cdot i^*(\sigma_{\mu^*}).$$

In turn, using the description of $i^*(\sigma_{\mu^*})$ from Proposition 2, (23) is rewritten as

$$(24) \quad m_{\lambda\mu} = \int_{G'} \sigma'_\lambda \cdot \left(\sum_{\nu} g_{\nu\mu^*} \sigma'_\nu \right) = \int_{G'} \sum_{\tau} \sum_{\nu} e_{\lambda\nu}^{\tau} g_{\nu\mu^*} \sigma'_\tau = g_{\lambda^\vee, \mu^*}$$

because only $\tau = (n, n-1, \dots, 1)$ and $\nu = \lambda^\vee$ give a nonzero contribution (note that for such τ and ν , we have $e_{\lambda\nu}^{\tau} = 1$).

7. Relations between the degrees of the ordinary and projective representations of the symmetric groups

For a partition μ , we set

$$(25) \quad \bar{f}^\mu := \prod_{x \in \mu} \frac{1}{h(x)},$$

where $h(x)$ is the hook-length of μ at $x = (i, j)$ defined by $h(x) = h(i, j) = \mu_i + \mu_j^\sim - i - j + 1$. If $|\mu| = m$ then $f^\mu := m! \bar{f}^\mu$ is the degree of the irreducible representation of S_m corresponding to μ . Equivalently, f^μ is the number of standard tableaux of shape μ , obtained by labeling the squares of the diagram of μ with the numbers $1, 2, \dots, m$. We refer to [F] for a detailed discussion of these facts.

For a strict partition λ , we set

$$(26) \quad \bar{g}^\lambda := \prod_{x \in S(\lambda)} \frac{1}{h(x)},$$

where $S(\lambda)$ is the shifted diagram associated with λ [M, p.255], and for each square $x \in S(\lambda)$ the hook-length $h(x)$ is defined to be the hook-length at x in the “double diagram” $(\lambda_1, \lambda_2, \dots | \lambda_1 - 1, \lambda_2 - 1, \dots)$, containing $S(\lambda)$. If $|\lambda| = m$, $g^\lambda := m! \bar{g}^\lambda$ is the number of shifted standard tableaux of shape $S(\lambda)$, obtained by labeling the squares of $S(\lambda)$ with the numbers $1, 2, \dots, m$ with strict increase along each row and down each column. The numbers g^λ also admit an interpretation as the degree of suitable projective representations of S_m . We refer to [H-H] for a detailed discussion of these results.

One has the following formulas, in terms of parts, for \bar{f}^μ [M, I.1 Example 1] and \bar{g}^λ [M, III.8 Example 12]:

$$(27) \quad \bar{f}^\mu = \frac{\prod_{i < j} (\mu_i - \mu_j - i + j)}{\prod_{i \geq 1} (\mu_i + n - i)!},$$

$$(28) \quad \bar{g}^\lambda = \frac{1}{\prod_{i \geq 1} \lambda_i!} \prod_{i < j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

Lemma 3. (i) Under the specialization $e_i := \frac{1}{i!}$, s_μ becomes \bar{f}^μ .

(ii) Under the specialization $Q_i := \frac{1}{i!}$, Q_λ becomes \bar{g}^λ .

(For assertion (i), see [M, I.3 Example 5]. Assertion (ii) stems from [DC-P, Proposition 6].)

Given a partition μ , we want to apply formulas (12) and (16), so we adopt the notation of §5. Also, we follow the notation of §6 associated with a sequence K . For such a sequence, we set

$$(29) \quad \bar{g}^K := \text{sgn}(w_K) \bar{g}^{<K>}.$$

From Lemma 3, (12), and (16), we get

Proposition 6. For a fixed partition μ , we have

$$(30) \quad \begin{aligned} 2^n \bar{f}^\mu &= \sum \bar{g}^{(a_{i_1}, \dots, a_{i_k})} \bar{g}^{A \# B \setminus (a_{i_1}, \dots, a_{i_k})} \\ &= \sum \bar{g}^{(c_{i_1}, \dots, c_{i_k})} \bar{g}^{C \# D \setminus (c_{i_1}, \dots, c_{i_k})}, \end{aligned}$$

where the sums are over all sequences $1 \leq i_1 < \dots < i_k \leq n$ for which $A \# B \setminus (a_{i_1}, \dots, a_{i_k})$ (resp. $C \# D \setminus (c_{i_1}, \dots, c_{i_k})$) is a sequence of different integers, and $k = 0, 1, \dots, n$.

For instance, for $\mu = (5^3 31^3) = (432|621)$, we get the equations:

$$\begin{aligned} 2^3 \bar{f}^{(5^3 31^3)} &= \bar{g}^{(654321)} - \bar{g}^{(5)} \bar{g}^{(64321)} + \bar{g}^{(4)} \bar{g}^{(65321)} - \bar{g}^{(3)} \bar{g}^{(65421)} - \bar{g}^{(54)} \bar{g}^{(6321)} \\ &\quad + \bar{g}^{(53)} \bar{g}^{(6421)} - \bar{g}^{(43)} \bar{g}^{(6521)} + \bar{g}^{(543)} \bar{g}^{(621)} \\ &= \bar{g}^{(32)} \bar{g}^{(7432)} + \bar{g}^{(732)} \bar{g}^{(432)}. \end{aligned}$$

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